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Entanglement and Berry Phase in a (3 × 3)-dimensional Yang-Baxter System

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Abstract A 9×9 unitary $\breve{R}$-matrix, solution of the Yang-Baxter Equation, is obtained in this paper. The entanglement properties of $\breve{R}$-matrix is investigated, and the arbitrary degree of entanglement for two-qutrit entangled states can be generated via $\breve{R}$-matrix acting on the standard basis. A Yang-Baxter Hamiltonian can be constructed from unitary $\breve{R}$-matrix. Then the geometric properties of this system is studied. The results showed that the Berry phase of this system can be represented under the framework of SU(2) algebra.

Keywords Entanglement · Berry phase · Yang-Baxter system

1 Introduction

Quantum entanglement (QE), the most surprising nonclassical property of quantum systems, plays a key role in quantum information and quantum computation processing [1–4]. Because of these applications, QE has become one of the most fascinating topics in quantum information and quantum computation. On the other hand, the geometrical phase [5], such as Berry phase (BP), is another important concept in quantum mechanics [6–10]. In recent years, a lot of works have been attributed to BP [11], because of its possible applications to quantum computation (the so-called geometric quantum computation) [12–15]. Such concern is motivated by the belief that geometric quantum gates should exhibit an intrinsic fault tolerance in the presence of some kind of external noise due to the geometric nature of the BP.

Yang-Baxter Equation (YBE) [16–20] was originated in solving quantum integrable models, but recently has been shown to have a deep connection with topological quantum computation and entanglement swapping [21–29]. In [30], the authors point out YBE can be

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tested in terms of quantum optics. In a very recently work [29], it is found that any pure two-qudit entangled state can be achieved by a universal Yang-Baxter Matrix assisted by local unitary transformations. However, the solution $\check{R}(x)$-matrix in [29] only dependent on one parameter. So we can't construct a Yang-Baxter Hamiltonian as in [27, 31]. In this paper, we obtain a time-dependent solution of YBE, $\check{R}(x,\varphi_1,\varphi_2)$. φ_1 and φ_2 are time-dependent, so we can construct Yang-Baxter Hamiltonian. Consequently, we can study entanglement properties and Berry phase for this system.

This paper is organized as follows: In Sect. 2, we present a 9×9 Yang-Baxter matrix. By means of negativity, we investigated the entanglement properties of $\check{R}(x,\varphi_1,\varphi_2)$-matrix. We show that the arbitrary degree of entanglement for two-qutrit entangled states can be generated via the unitary matrix $\check{R}(\theta,\varphi_1,\varphi_2)$-matrix acting on the standard basis. In Sect. 3, we construct a Hamiltonian from the unitary $\check{R}(\theta,\varphi_1,\varphi_2)$-matrix. The Berry phase of the system is investigated, and the results showed that the Berry phase of this system can be represented under the framework of SU(2) algebra. The summary is made in the last section.

2 Unitary Solution of Yang-Baxter Equation and Its Entanglement Properties

The usual YBE takes the form,

$$\check{R}_i(x)\check{R}_{i+1}(xy)\check{R}_i(y)=\check{R}_{i+1}(y)\check{R}_i(xy)\check{R}_{i+1}(x) \tag{1}$$

The spectral parameters x and y which are related with the one-dimensional momentum play an important role in some typical models [16, 17]. The asymptotic behavior of $\check{R}(x,\varphi_1,\varphi_2)$ is x-independent, i.e. $\lim \check{R}_{i,i+1}(x,\varphi_1,\varphi_2)\propto b_i$, where b_i are braiding operators, which satisfy the braiding relations,

$$\begin{cases} b_i b_{i+1} b_i = b_{i+1} b_i b_{i+1} & 1\le i<n-2 \\ b_i b_j = b_j b_i & |i-j|\ge 2 \end{cases} \tag{2}$$

where the notation $b_i\equiv b_{i,i+1}$ is used, $b_{i,i+1}$ represents $1_1\otimes 1_2\otimes 1_3\otimes\cdots\otimes S_{i,i+1}\otimes\cdots\otimes 1_n$, and 1_j is the unit matrix of the j-th particle.

As is known, Hecke algebras are intimately connected with braiding groups. In fact, braid algebra is subalgebra of Hecke algebra. And we can construct a representation of braid algebra from Hecke algebra. A unitary solution of YBE can also be constructed from a representation of Hecke algebra. Let us review Yang-Baxterization [31–33] of Hecke algebra. M_i, a Hermitian matrix (i.e. $M_i^\dagger = M_i$), satisfies the Hecke algebraic relations: $M_iM_{i+1}M_i+gM_i=M_{i+1}M_iM_{i+1}+gM_{i+1}$ and $M_i^2=\alpha M_i+\beta I_i$. For convenience, we set $\alpha=1$ and $\beta=g=2$. let the unitary Yang-Baxter matrix take the form,

$$\check{R}_i(x)=\rho(x)[\mathbf{1}_i+F(x)M_i] \tag{3}$$

Substituting (3) into (1), one has $F(x)+F(y)+F(x)F(y)=[1+2F(x)F(y)]F(xy)$. The unitary condition (i.e., $\check{R}_i^\dagger(x)=\check{R}_i^{-1}(x)=\check{R}_i(x^{-1})$) can be tenable only on condition that $F(x)+F(x^{-1})+F(x)F(x^{-1})=0$ and $\rho(x)\rho(x^{-1})[1+2F(x)F(x^{-1})]=0$. In addition, the initial condition $\check{R}_i(x=1)=I_i$ yields $F(x=1)=0$ and $\rho(x=1)=1$. Taking account into these conditions, we obtain a set solutions of $F(x)$ and $\rho(x)$,

$$\rho(x)=\frac{2x+x^{-1}}{3},\qquad F(x)=-\frac{x-x^{-1}}{2x+x^{-1}}.$$

In this paper, we choose basis $\{|11\rangle, |10\rangle, |01\rangle, |1-1\rangle, |00\rangle, |-11\rangle, |0-1\rangle, |-10\rangle, |-1-1\rangle\}$ as the standard basis. Based on calculation, a 9×9 matrix M which satisfies the Hecke algebraic relations is realized as,

$$\begin{aligned} M_{cd}^{ab} &= q_1\delta_{ab1}|_{a\neq c\neq d} + q_2\delta_{ab0}|_{a\neq c\neq d} \\ &\quad + Q^{-1}\delta_{ab-1}|_{a\neq c\neq d} + q_1^{-1}\delta_{cd1}|_{c\neq a\neq b} \\ &\quad + q_2^{-1}\delta_{cd0}|_{c\neq a\neq b} + Q\delta_{cd-1}|_{c\neq a\neq b} \\ &\quad + \delta_{ad}\delta_{bc}\,|_{a\neq b} \end{aligned} \tag{4}$$

where $q_1 = e^{i\varphi_1}$, $q_2 = e^{i\varphi_2}$ and $Q = q_1 q_2$, with the parameters φ_1 and φ_2 both are real. The denotes $M_{cd}^{ab} \equiv M_{ab,cd}$ are used. The denote $\delta_{abc} = 1$, if and only if $a = b = c$; otherwise, the denote $\delta_{abc} = 0$. This solution is not equivalent to the solution in [31]. Substituting (4) into (3), the unitary solution of YBE can be obtained as following,

$$\breve{R}(x,\varphi_1,\varphi_2)_{cd}^{ab} = \rho(x)[\delta_{abcd} + F(x)M_{cd}^{ab}] \tag{5}$$

The matrix form of $\breve{R}_i(x,\varphi_1,\varphi_2)$ can be recast as,

$$\breve{R}_i(x,q_i) = \frac{1}{3}\begin{pmatrix} b & 0 & 0 & 0 & 0 & 0 & aq_1 & aq_1 & 0 \\ 0 & b & a & 0 & 0 & 0 & 0 & 0 & aQ \\ 0 & a & b & 0 & 0 & 0 & 0 & 0 & aQ \\ 0 & 0 & 0 & b & \frac{a}{q_2} & a & 0 & 0 & 0 \\ 0 & 0 & 0 & aq_2 & b & aq_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a & \frac{a}{q_2} & b & 0 & 0 & 0 \\ \frac{a}{q_1} & 0 & 0 & 0 & 0 & 0 & b & a & 0 \\ \frac{a}{q_1} & 0 & 0 & 0 & 0 & 0 & a & b & 0 \\ 0 & \frac{a}{Q} & \frac{a}{Q} & 0 & 0 & 0 & 0 & 0 & b \end{pmatrix} \tag{6}$$

where $a = x^{-1} - x$, $b = 2x + x^{-1}$. The Gell-Mann matrices, a basis for the Lie algebra SU(3) [34], λ_u satisfy $[I_\lambda, I_\mu] = i f_{\lambda\mu\nu} I_\nu$ $(\lambda,\mu,\nu = 1,\ldots,8)$, where $I_\mu = \frac{1}{2}\lambda_\mu$. For the later convenience, we denote I_λ by, $I_\pm = I_1 \pm i I_2$, $V_\pm = I_4 \mp i I_5$, $U_\pm = I_6 \pm i I_7$, $Y = \frac{2}{\sqrt{3}} I_8$. In this work, we get rise to three sets of realization of SU(3) as:

$$\begin{cases} I_\pm^{(1)} = I_1^\pm I_2^\mp, \qquad U_\pm^{(1)} = U_1^\pm V_2^\mp, \qquad V_\pm^{(1)} = V_1^\pm U_2^\mp, \\ I_3^{(1)} = \frac{1}{3}(I_1^3 - I_2^3) + \frac{1}{2}(I_1^3 Y_2 - Y_1 I_2^3), \\ Y^{(1)} = \frac{1}{3}(Y_1 + Y_2) - \frac{2}{3} I_1^3 I_2^3 - \frac{1}{2} Y_1 Y_2; \end{cases}$$

$$\begin{cases} I_\pm^{(2)} = U_1^\pm U_2^\mp, \qquad U_\pm^{(2)} = V_1^\pm I_2^\mp, \qquad V_\pm^{(2)} = I_1^\pm V_2^\mp, \\ I_3^{(2)} = \frac{1}{2}[-\frac{1}{3}(I_1^3 - I_2^3) + \frac{1}{2}(Y_1 - Y_2) + I_1^3 Y_2 - Y_1 I_2^3], \\ Y^{(2)} = -[\frac{1}{3}(I_1^3 + I_2^3) + \frac{1}{6}(Y_1 + Y_2) + \frac{2}{3} I_1^3 I_2^3 + \frac{1}{2} Y_1 Y_2]; \end{cases}$$

$$\begin{cases} I_{\pm}^{(3)} = V_1^{\pm} V_2^{\mp}, \qquad U_{\pm}^{(3)} = I_1^{\pm} U_2^{\mp}, \qquad V_{\pm}^{(3)} = U_1^{\pm} I_2^{\mp}, \\ I_3^{(3)} = \frac{1}{2}[-\frac{1}{3}(I_1^3 - I_2^3) - \frac{1}{2}(Y_1 - Y_2) + I_1^3 Y_2 - Y_1 I_2^3], \\ Y^{(3)} = \frac{1}{3}(I_1^3 + I_2^3) - \frac{1}{6}(Y_1 + Y_2) - \frac{2}{3} I_1^3 I_2^3 - \frac{1}{2} Y_1 Y_2. \end{cases}$$

We denote $I_{\pm}^{(k)} = I_1^{(k)} \pm i I_2^{(k)}$, $V_{\pm}^{(k)} = I_4^{(k)} \mp i I_5^{(k)}$, $U_{\pm}^{(k)} = I_6^{(k)} \pm i I_7^{(k)}$, $Y^{(k)} = \frac{2}{\sqrt{3}} I_8^{(k)}$ $(k = 1, 2, 3)$. These realizations satisfy the commutation relation $[I_\lambda^{(i)}, I_\mu^{(j)}] = i\delta_{ij} f_{\lambda\mu\nu} I_\nu^{(i)}$ $(\lambda, \mu, \nu = 1, \ldots, 8; i, j = 1, 2, 3)$. So the whole tensor space $C^3 \otimes C^3$ is completely decomposed. In addition, each block of $\check{R}$-matrix can be represented by fundamental representation of SU(3) algebra, i.e. $C^3 \otimes C^3 = C^3 \oplus C^3 \oplus C^3$.

For i-th and $(i+1)$-th lattices, $\check{R}$-matrix can be expressed in terms of above operators,

$$\begin{aligned} \check{R}(\theta, \varphi_1, \varphi_2) = \frac{1}{3} a [& I_+^{(1)} + I_-^{(1)} + Q(V_-^{(1)} + U_+^{(1)}) \\ & + Q^{-1}(U_-^{(1)} + V_+^{(1)}) + I_+^{(2)} + I_-^{(2)} \\ & + q_1(V_+^{(2)} + U_-^{(2)}) + q_1^{-1}(V_-^{(2)} + U_+^{(2)}) \\ & + I_+^{(3)} + I_-^{(3)} + q_2(V_+^{(3)} + U_-^{(3)}) \\ & + q_2^{-1}(V_-^{(3)} + U_+^{(3)})] + \frac{b}{3}(I \otimes I). \end{aligned}$$

We can introduce a new variable with $x = e^{i\theta}$, and θ may be related with entanglement degree. When one acts $\check{R}(\theta, \varphi_1, \varphi_2)$ on the separable state $|mn\rangle$, he yields the following family of states $|\psi\rangle_{mn} = \sum_{ij=11}^{-1-1} \check{R}_{mn}^{ij} |mn\rangle$ $(m, n = 1, 0, -1)$. For example, if $m = 1$ and $n = 1$, $|\psi\rangle_{11} = \frac{1}{3}(b|11\rangle + a q_1^{-1}|0-1\rangle + a q_1^{-1}|-10\rangle)$. By means of negativity [35–37], we study these entangled states. The negativity for two qutrits is given by,

$$N(\rho) \equiv \frac{\|\rho^{T_A}\| - 1}{2}, \tag{7}$$

where $\|\rho^{T_A}\|$ denotes the trace norm of ρ^{T_A}, and ρ^{T_A} denotes the partial transpose of the bipartite state ρ, i.e., $(\rho)_{j_A j_B}^{i_A i_B} = (\rho^{T_A})_{i_A j_B}^{j_A i_B}$. In fact, $N(\rho)$ corresponds to the absolute value of the sum of negative eigenvalues of ρ^{T_A}, and negativity vanishes for unentangled states [36]. Then we can obtain the negativity of the state $|\psi\rangle_{11}$ as

$$N(\theta) = \frac{4}{9}(\sin^2\theta + |\sin\theta|\sqrt{1 + 8\cos^2\theta}). \tag{8}$$

When $|a| = |b|$, namely $x = e^{i\frac{\pi}{3}}$, the state $|\psi\rangle_{11}$ becomes the maximally entangled state of two qutrits as $|\psi\rangle_{11} = \frac{1}{\sqrt{3}}(e^{i\frac{\pi}{6}}|11\rangle - i q_1^{-1}|0-1\rangle - i q_1^{-1}|-10\rangle)$. In general, if one acts the unitary Yang-Baxter matrix $\check{R}(x)$ on the basis $\{|11\rangle, |10\rangle, |01\rangle, |1-1\rangle, |00\rangle, |-11\rangle, |0-1\rangle, |-10\rangle, |-1-1\rangle\}$, he will obtain the same negativity as (8). It is easy to check that the negativity ranges from 0 to 1 when the parameter θ runs from 0 to π. But for $\theta \in [0, \pi]$, the negativity is not a monotonic function of θ. And when $x = e^{i\frac{\pi}{3}}$, he will generate nine complete and orthogonal maximally entangled states for two qutrits. The QE doesn't dependent on the parameters φ_1 and φ_2. So one can verify that parameter φ_1 and φ_2 may be absorbed into a local operation. Base on numerical calculation, the universality of YBE is proved by Chen et al. [29]. This unitary solution of YBE can generate entangled states, this solution may be a universal quantum gate.

3 Yang-Baxter Hamiltonian and BP

A Hamiltonian of the Yang-Baxter system can be constructed from the $\breve{R}(\theta, \varphi_1, \varphi_2)$-matrix. As shown in [27], the Hamiltonian is obtained through the Schrödinger evolution of the entangled states. Let the parameters φ_i be time-dependent as $\varphi_i = \omega_i t$. The Hamiltonian reads,

$$\hat{H} = i\hbar \frac{\partial \breve{R}(\theta, \varphi_1, \varphi_2)}{\partial t} \breve{R}^\dagger(\theta, \varphi_1, \varphi_2) = \bigoplus_{k=1}^{3} H^{(k)}, \tag{9}$$

where the superscript k denotes the k-th subsystem. The k-th subsystem's Hamiltonian $H^{(k)}$ can be obtained as following,

$$H^{(1)} = C(1)\left[\frac{\sqrt{2}}{6}\sin\theta(I_+^{(1)} + I_-^{(1)}) + \frac{\sqrt{2}}{2}\sin\theta Y^{(1)} - \frac{\sqrt{2}}{12} i b^* Q(V_-^{(1)} + U_+^{(1)}) + \frac{\sqrt{2}}{12} i b Q^{-1}(V_+^{(1)} + U_-^{(1)})\right] \tag{10}$$

$$H^{(2)} = C(2)\left[-\frac{\sqrt{2}}{6}\sin\theta(I_+^{(2)} + I_-^{(2)}) - \frac{\sqrt{2}}{2}\sin\theta Y^{(2)} + \frac{\sqrt{2}}{12} i b^* q_1^{-1}(U_+^{(2)} + V_-^{(2)}) - \frac{\sqrt{2}}{12} i b q_1(V_+^{(2)} + U_-^{(2)})\right] \tag{11}$$

$$H^{(3)} = C(3)\left[-\frac{\sqrt{2}}{6}\sin\theta(I_+^{(3)} + I_-^{(3)}) - \frac{\sqrt{2}}{2}\sin\theta Y^{(3)} + \frac{\sqrt{2}}{12} i b^* q_2^{-1}(U_+^{(3)} + V_-^{(3)}) - \frac{\sqrt{2}}{12} i b q_2(V_+^{(3)} + U_-^{(3)})\right] \tag{12}$$

where $C(1) = -\frac{4\sqrt{2}\hbar\Omega\sin\theta}{3}$, $C(2) = -\frac{4\sqrt{2}\hbar\omega_1\sin\theta}{3}$, $C(3) = -\frac{4\sqrt{2}\hbar\omega_2\sin\theta}{3}$ and $\Omega \equiv \omega_1 + \omega_2$. In terms of $I_\lambda^{(k)}$ $(\lambda = 1, 2, \ldots, 8; k = 1, 2, 3)$, the Hamiltonian can be recast as following,

$$H^{(k)} = C(k)\sum_{\lambda=1}^{8} B_\lambda^{(k)} I_\lambda^{(k)}. \tag{13}$$

Compare (10), (11), (12) with (13), one can obtain $B_\lambda^{(k)}$ as following,

$$\begin{cases} B_1^{(1)} = \frac{\sqrt{2}}{3}\sin\theta; \qquad B_2^{(1)} = B_3^{(1)} = 0 \\ B_4^{(1)} = -\frac{\sqrt{2}}{6}\sin\theta\cos\omega(1)t + \frac{\sqrt{2}}{2}\cos\theta\sin\omega(1)t \\ B_5^{(1)} = \frac{\sqrt{2}}{6}\sin\theta\sin\omega(1)t + \frac{\sqrt{2}}{2}\cos\theta\cos\omega(1)t \\ B_6^{(1)} = -\frac{\sqrt{2}}{6}\sin\theta\cos\omega(1)t + \frac{\sqrt{2}}{2}\cos\theta\sin\omega(1)t \\ B_7^{(1)} = \frac{\sqrt{2}}{6}\sin\theta\sin\omega(1)t + \frac{\sqrt{2}}{2}\cos\theta\cos\omega(1)t \\ B_8^{(1)} = \frac{\sqrt{2}}{2}\sin\theta \end{cases}$$

$$\begin{cases} B_1^{(i)} = -\frac{\sqrt{2}}{3}\sin\theta; \qquad B_2^{(i)} = B_3^{(i)} = 0 \\ B_4^{(i)} = \frac{\sqrt{2}}{6}\sin\theta\cos\omega(i)t + \frac{\sqrt{2}}{2}\cos\theta\sin\omega(i)t \\ B_5^{(i)} = \frac{\sqrt{2}}{6}\sin\theta\sin\omega(i)t - \frac{\sqrt{2}}{2}\cos\theta\cos\omega(i)t \\ B_6^{(i)} = \frac{\sqrt{2}}{6}\sin\theta\cos\omega(i)t + \frac{\sqrt{2}}{2}\cos\theta\sin\omega(i)t \\ B_7^{(i)} = \frac{\sqrt{2}}{6}\sin\theta\sin\omega(i)t - \frac{\sqrt{2}}{2}\cos\theta\cos\omega(i)t \\ B_8^{(i)} = -\frac{\sqrt{2}}{2}\sin\theta \end{cases}$$

where $i = 2, 3$. The denotes $\omega(1) \equiv \Omega$, $\omega(2) \equiv \omega_1$ and $\omega(3) \equiv \omega_2$ are used. The Hamiltonian for the k-th subsystem, $H(\mathbf{B}(t)^{(k)})^{(k)}$, depends on the parameters $B_\lambda^{(k)}$ $(\lambda = 1, 2, \ldots, 8)$, which are the components of a vector $\mathbf{B}^{(k)}$. And $\mathbf{B}^{(k)}$ are a set of time-varying parameters controlling the k-th subsystem's Hamiltonian. After time $T^{(k)}$, Hamiltonian returns to its original form, i.e. $H(\mathbf{B}(0))^{(k)} = H(\mathbf{B}(T^{(k)}))^{(k)}$. According to this, one can easily verify periods of the subsystems are $T^{(1)} = 2\pi/\Omega$, $T^{(2)} = 2\pi/\omega_1$ and $T^{(3)} = 2\pi/\omega_2$. The eigenstates of the first subsystem are found to be,

$$|E_+^{(1)}\rangle = N_+^{(1)}[f_+^{(1)}(|10\rangle + |01\rangle) + e^{-i\Omega t}|-1-1\rangle]$$
$$|E_0^{(1)}\rangle = \frac{1}{\sqrt{2}}(-|10\rangle + |01\rangle)$$
$$|E_-^{(1)}\rangle = N_-^{(1)}(f_-^{(1)}(|10\rangle + |01\rangle) + e^{-i\Omega t}|-1-1\rangle)$$

with the corresponding eigenvalues $E_+^{(1)} = \frac{2\sqrt{2}}{3}\hbar\Omega\sin\theta$, $E_0^{(1)} = 0$ and $E_-^{(1)} = -\frac{2\sqrt{2}}{3}\hbar\Omega\sin\theta$. Where $N_\pm^{(1)} = \sqrt{\frac{3\pm 2\sqrt{2}\sin\theta}{6}}$ and $f_\pm^{(1)} = \frac{4\sin\theta\mp 3\sqrt{2}}{2ib(\theta)}$. For the second and the third subsystems, the eigenstates are found to be,

$$|E_+^{(2)}\rangle = N_+^{(2)}[f_+^{(2)}|11\rangle + e^{-i\omega_1 t}(|0-1\rangle + |-10\rangle)]$$

$$|E_0^{(2)}\rangle = \frac{1}{\sqrt{2}}(-|0-1\rangle + |-10\rangle)$$

$$|E_-^{(2)}\rangle = N_-^{(2)}[f_-^{(2)}|11\rangle + e^{-i\omega_1 t}(|0-1\rangle + |-10\rangle)].$$

$$|E_+^{(3)}\rangle = N_+^{(3)}[f_+^{(3)}|00\rangle + e^{-i\omega_2 t}(|1-1\rangle + |-11\rangle)]$$

$$|E_0^{(3)}\rangle = \frac{1}{\sqrt{2}}(-|1-1\rangle + |-11\rangle)$$

$$|E_-^{(3)}\rangle = N_-^{(3)}[f_-^{(3)}|00\rangle + e^{-i\omega_2 t}(|1-1\rangle + |-11\rangle)].$$

with the corresponding eigenvalues $E_+^{(i)} = \frac{2\sqrt{2}}{3}\hbar\omega(i)\sin\theta$, $E_0^{(i)} = 0$ and $E_-^{(i)} = -\frac{2\sqrt{2}}{3}\hbar\omega(i)\sin\theta$. Where $N_\pm^{(i)} = \sqrt{\frac{3\pm 2\sqrt{2}\sin\theta}{12}}$ and $f_\pm^{(i)} = \frac{4\sin\theta \mp 3\sqrt{2}}{ib^*(\theta)}$. According to the definition of the BP [5], when the parameter $\mathbf{B}^{(k)}$ is slowly changed around a circuit on the sphere of direction, then at the end of circuit, the eigenstates $|E_\alpha^k\rangle$ ($\alpha = +, 0, -$) evolves adiabatically from 0 to $T^{(k)}$, the BP accumulated by the states $|E_\alpha^k\rangle$ are,

$$\gamma_\alpha^{(k)} = i\int_0^{T^{(k)}} \langle E_\alpha^{(k)}|\frac{\partial}{\partial t}|E_\alpha^{(k)}\rangle dt. \tag{14}$$

Substitute these eigenstates into (14), one can obtain the BP for these eigenstates (all phases are defined modulo 2π throughout this paper),

$$\begin{cases} \gamma_+^k = -(\frac{1}{2} - \frac{\sqrt{2}}{3}\sin\theta)2\pi \\ \gamma_0^k = 0 \\ \gamma_-^k = (\frac{1}{2} - \frac{\sqrt{2}}{3}\sin\theta)2\pi. \end{cases} \tag{15}$$

In fact, the BP of this system can be represented under the framework of SU(2) algebra. First, we can introduce three sets SU(2) realizations in terms of three new sets of operators,

$$\begin{cases} S_+^{(k)} = \frac{1}{\sqrt{2}}(V_-^{(k)} + U_+^{(k)}) \\ S_-^{(k)} = \frac{1}{\sqrt{2}}(V_+^{(k)} + U_-^{(k)}) \\ S_3^{(k)} = \frac{3}{4}Y^{(k)} + \frac{1}{4}(I_+^{(k)} + I_-^{(k)}). \end{cases} \tag{16}$$

They satisfy the algebraic relations of SU(2) group: $[S_+^{(i)}, S_-^{(j)}] = 2\delta_{ij}S_3^{(i)}$, $[S_3^{(i)}, S_\pm^{(j)}] = \pm\delta_{ij}S_\pm^{(i)}$, $(S_\pm^{(i)})^2 = 0$ ($i, j = 1, 2, 3$), with $S_\pm^{(k)} = S_1^{(k)} \pm iS_2^{(k)}$ ($k = 1, 2, 3$). By the way, their second-order Casimir operators are $\mathscr{J}^{(k)} = \frac{1}{2}(S_+^{(k)}S_-^{(k)} + S_-^{(k)}S_+^{(k)}) + (S_3^{(k)})^2$. One can verify that the eigenvalues of $\mathscr{J}^{(k)}$ are $\frac{1}{2}(\frac{1}{2}+1) = \frac{3}{4}$ and $0(0+1) = 0$ which correspond to spin-$1/2$ system and spin-0 system.

When one substitutes these realizations into (10)–(12), he can recast Hamiltonian of the subsystems in terms of SU(2),

$$H^{(1)} = C(1)\left[\frac{1}{2}(B_-^{(1)}S_+^{(1)} + B_+^{(1)}S_-^{(1)}) + B_3^{(1)}S_3^{(1)}\right]$$

$$H^{(2)} = C(2)\left[\frac{1}{2}(B_-^{(2)}S_+^{(2)} + B_+^{(2)}S_-^{(2)}) + B_3^{(2)}S_3^{(2)}\right] \tag{17}$$

$$H^{(3)} = C(3)\left[\frac{1}{2}(B_-^{(3)} S_+^{(1)} + B_+^{(3)} S_-^{(3)}) + B_3^{(3)} S_3^{(3)}\right]$$

where $B_-^{(1)} = (B_+^{(1)})^* = -\frac{1}{3} i b^* e^{i\Omega t}$, $B_3^{(1)} = \frac{2\sqrt{2}}{3}\sin\theta$, $B_-^{(i)} = (B_+^{(i)})^* = \frac{1}{3} i b^* e^{-i\omega(i)t}$ and $B_3^{(i)} = -\frac{2\sqrt{2}}{3}\sin\theta$ $(i = 1; 2)$. $\omega(i)$ and C(3) are defined below (13). So we can say the whole system equivalent to three spin-$\frac{1}{2}$ subsystems and three spin-0 subsystems. In fact, we can introduce a time-independent 9×9 orthogonal matrix O (see Appendix A). By means of O, the whole system's Hamiltonian $\hat{H}$ and Casimir operators $\mathscr{J}^{(k)}$ are transformed into block-diagonal matrices, i.e. $\tilde{\hat{H}} = O\hat{H}O^T$ and $\tilde{\mathscr{J}}^{(k)} = O\mathscr{J}^{(k)}O^T$ are block-diagonal matrices, where O^T denotes the transpose of matrix O. For the subsystem 1, from (22) we can get its Hamiltonian $\tilde{H}^{(1)} = H_{\frac{1}{2}}^{(1)} \oplus H_0^{(1)}$. For $H_0^{(1)}$, the eigenvalue of Casimir operator $\mathscr{J}^{(1)}$ is 0, and the BP is 0. So we can say the subsystem Hamiltonian $H_0^{(1)}$ is equivalent to a spin-0 subsystem. For $H_{\frac{1}{2}}^{(1)}$, one can introduce two transformations, $\cos\alpha = \frac{2\sqrt{2}}{3}\sin\theta$ and $\cos\beta = \frac{-\sin\theta\cos\Omega t + 3\cos\theta\sin\Omega t}{\sqrt{9-8\sin^2\theta}}$, with $\alpha \in (\arccos\frac{2\sqrt{2}}{3}, \arccos -\frac{2\sqrt{2}}{3})$ and $\beta \in [0, 2\pi]$. α is time-independent, and β is time-dependent. By means of this transformation, the Hamiltonian $H_{\frac{1}{2}}^{(1)}$ can be recast as $H_{\frac{1}{2}}^{(1)} = C(1)(\sin\alpha\cos\beta S_1 + \sin\alpha\sin\beta S_2 + \cos\alpha S_3)$. We substitute these transformations into (15), the BP of the subsystem $H_{\frac{1}{2}}^{(1)}$ can be recast as,

$$\gamma_\pm^{(1)} = \mp\pi(1-\cos\alpha) = \mp\Omega(C)/2 \tag{18}$$

where $\Omega(C) = 2\pi(1-\cos\alpha)$ is the familiar solid angle enclosed by the loop on the Bloch sphere, and the parameter α comes from θ which comes from the Yang-Baxterization of the unitary braiding operator. So the BP dependent on spectral parameter. Under the new basis (22), the eigenstates $|E_\pm^{(1)}\rangle$ can be recast as following (we neglected the global phase factor),

$$|E_+^{(1)}\rangle = -e^{-i\beta}\sin\frac{\alpha}{2}|1\rangle + \cos\frac{\alpha}{2}|2\rangle, \tag{19}$$

$$|E_-^{(1)}\rangle = \cos\frac{\alpha}{2}|1\rangle + e^{i\beta}\sin\frac{\alpha}{2}|2\rangle, \tag{20}$$

where $|1\rangle = O\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ and $|2\rangle = O|-1-1\rangle$. As is known to all, they are spin coherent states. By means of (24), the states $|E_\pm^{(1)}\rangle$ can be recast as following,

$$\begin{aligned}|E_+^{(1)}\rangle &= \exp[\zeta\tilde{S}_+^{(1)} - \zeta^*\tilde{S}_-^{(1)}]|2\rangle,\\ |E_-^{(1)}\rangle &= \exp[\zeta\tilde{S}_+^{(1)} - \zeta^*\tilde{S}_-^{(1)}]|1\rangle,\end{aligned} \tag{21}$$

where $\zeta = e^{-i\beta}\alpha/2$. BP for spin coherent states has been investigated in [38]. So we can say the subsystem $H_{\frac{1}{2}}^{(1)}$ is equivalent to a spin-$\frac{1}{2}$ subsystem. By means of the same method, the Berry phases for subsystem 2 and 3 may be obtained, $\gamma_\pm^{(k)} = \mp\pi(1-\cos\alpha) = \mp\Omega(C)/2$ and $\gamma_0^{(k)} = 0$. The whole system is equivalent to three spin$-1/2$ subsystems and three spin-0 subsystems. This Yang-Baxter Hamiltonian system is equivalent to the Hamiltonian in [31].

4 Summary

In this paper, we have presented a 9×9 M-matrix which satisfies the Hecke algebraic relations and derived a unitary $\breve{R}(\theta, \varphi_1, \varphi_2)$-matrix via Yang-Baxterization of the M-matrix.

In the following, we show that the arbitrary degree of entanglement for two-qutrit entangled states can be generated via the unitary $\breve{R}(\theta,\varphi_1,\varphi_2)$ matrix acting on the standard basis. Then the evolution of the Yang-Baxter system is explored by constructing a Hamiltonian from the unitary $\breve{R}(\theta,\varphi_1,\varphi_2)$-matrix. In addition, the BP of the system is investigated. By means of decomposition of the tensor product, the Berry phase of the whole system is explained. The whole system is equivalent to three spin-$\frac{1}{2}$ subsystems and three spin-0 subsystems. Berry phase of this system is represented under the framework of SU(2) algebra.

Thermal entanglement in multi-body system is an interesting and nature type of QE, so it is a good challenge to study the thermal entanglement in multi-body system. Yang-Baxter Equation is an important tool in this domain.

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Appendix A: Block-Diagonalize $\hat{H}$ and $\mathscr{J}^{(k)}$

The time-independent 9×9 orthogonal matrix O reads,

$$O=\begin{pmatrix}
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0\\
0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0
\end{pmatrix}$$

The orthogonal matrix O satisfies the relation $OO^T=O^TO=I_{9\times 9}$, where O^T denotes the transpose of matrix O.

The orthogonal matrix O transforms the standard basis $\{|11\rangle,|10\rangle,|01\rangle,|1-1\rangle,|00\rangle,|-11\rangle,|0-1\rangle,|-10\rangle,|-1-1\rangle\}$ into a new set of basis. The relations of new basis and old basis are,

$$\begin{cases}
|1\rangle=O\frac{1}{\sqrt{2}}(|10\rangle+|01\rangle)\\
|2\rangle=O|-1-1\rangle\\
|3\rangle=O\frac{1}{\sqrt{2}}(-|10\rangle+|01\rangle)
\end{cases}$$
$$\begin{cases}
|4\rangle=O|11\rangle\\
|5\rangle=O\frac{1}{\sqrt{2}}(|0-1\rangle+|-10\rangle)\\
|6\rangle=O\frac{1}{\sqrt{2}}(-|0-1\rangle+|-10\rangle)
\end{cases}\tag{22}$$
$$\begin{cases}
|7\rangle=O|11\rangle\\
|8\rangle=O\frac{1}{\sqrt{2}}(|1-1\rangle+|-11\rangle)\\
|9\rangle=O\frac{1}{\sqrt{2}}(-|1-1\rangle+|-11\rangle)
\end{cases}$$

$\{|1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle, |6\rangle, |7\rangle, |8\rangle, |9\rangle\}$ are a set of new basis. By means of this set basis, the Hamiltonian $\hat{H}$ can be recast as block-diagonally form,

$$\begin{aligned}\tilde{\hat{H}} &= O\hat{H}O^T \\ &= \mathrm{diag}\{H_{\frac{1}{2}}^{(1)}, H_0^{(1)}, H_{\frac{1}{2}}^{(2)}, H_0^{(2)}, H_{\frac{1}{2}}^{(3)}, H_0^{(3)}\} \\ &= \bigoplus_{k=1}^{3} \tilde{H}^{(k)},\end{aligned} \tag{23}$$

where $\tilde{H}^{(k)} = H_{\frac{1}{2}}^{(k)} \oplus H_0^{(k)}$, $H_{\frac{1}{2}}^{(k)}$'s are 2×2 matrix, and $H_0^{(k)}$ are 1×1 matrix with $H_0^{(k)} = (0)$. Under the new basis, the (3×3)-dimension matrix is decomposed into six blocks.

Three sets of SU(2) realizations (16) can be recast as,

$$\begin{aligned}\tilde{S}_+^{(1)} &= |1\rangle\langle 2|, \qquad \tilde{S}_-^{(1)} = |2\rangle\langle 1|, \\ \tilde{S}_3^{(1)} &= \frac{1}{2}(|1\rangle\langle 1| - |2\rangle\langle 2|);\end{aligned} \tag{24}$$

$$\begin{aligned}\tilde{S}_+^{(2)} &= |4\rangle\langle 5|, \qquad \tilde{S}_-^{(2)} = |5\rangle\langle 4|, \\ \tilde{S}_3^{(2)} &= \frac{1}{2}(|4\rangle\langle 4| - |5\rangle\langle 5|);\end{aligned} \tag{25}$$

$$\begin{aligned}\tilde{S}_+^{(3)} &= |7\rangle\langle 8|, \qquad \tilde{S}_-^{(2)} = |8\rangle\langle 7|, \\ \tilde{S}_3^{(2)} &= \frac{1}{2}(|7\rangle\langle 7| - |8\rangle\langle 8|).\end{aligned} \tag{26}$$

The second-order Casimir operators are $\tilde{\mathscr{J}}^{(1)} = \frac{3}{4}(|1\rangle\langle 1| + |2\rangle\langle 2|)$, $\tilde{\mathscr{J}}^{(2)} = \frac{3}{4}(|4\rangle\langle 4| + |5\rangle\langle 5|)$, $\tilde{\mathscr{J}}^{(3)} = \frac{3}{4}(|7\rangle\langle 7| + |8\rangle\langle 8|)$.

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